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Odd Viscosity

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When time reversal is broken, the viscosity tensor can have a nonvanishing odd part. In two dimensions, and only then, such odd viscosity is compatible with isotropy. Elementary and basic features of odd viscosity are examined by considering solutions of the wave and Navier–Stokes equations for hypothetical fluids where the stress is dominated by odd viscosity.

KEY WORDS: Nondissipative viscosity; viscosity waves; Navier-Stokes.

1. INTRODUCTION AND OVERVIEW

Normally, one associates viscosity with dissipation. However, as the viscosity is, in general, a tensor, this need not be the case since the antisymmetric part of a tensor is not associated with dissipation. We call the antisymmetric part odd. It must vanish, by Onsager relation, if time reversal holds. It must also vanish in three dimensions if the tensor is isotropic. But, in two dimensions odd viscosity is compatible with isotropy.

It is conceivable that that odd viscosity does not vanish for many system where time reversal is broken either spontaneously or by external fields. But, I know of only two systems for which there are theoretical studies of the odd viscosity and none for which it has been studied experimentally. In superfluid He^3, time reversal and isotropy can spontaneously break and the odd viscosity has three independent components.[9] As far as I know there is no estimate for their magnitudes. In the two dimensional quantum Hall fluid time reversal is broken by an external magnetic field. In the case of a full Landau level the dissipative viscosity vanishes. The odd viscosity, η^a, has been calculated for non interacting electrons in[4] for the lowest

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Landau level. Using results of,[5] the odd viscosity for any integral filling factor n is:

$$\eta^a(n) = \frac{n^2}{2}\frac{eB}{4c}, \qquad n = 1,... \tag{1}$$

e is the charge of the electron, c the velocity of light. It is an amusing coincidence of the cgs units that the *kinematic viscosity* of the electron gas, at integer fillings,

$$\left(\frac{\eta^a}{\rho_e}(n)\right) = n\frac{h}{8m_e} \tag{2}$$

is close to n in cgs units.

Significant odd viscosity can be responsible for odd properties and we shall illustrate these by considering the wave and Navier–Stokes equations for hypothetical media where the odd viscosity dominates. A good example for such a peculiar property is the following: Consider a small, slowly rotating circular cylinder (so that the Reynolds number is small) in an fluid which has both dissipating and and odd viscosity. The dissipative viscosity applies a torque which resists the rotation. This agrees with common intuition. The odd viscosity leads to *radial* pressure on the cylinder which is proportional to the rate of rotation (and the coefficient of odd viscosity, of course). Reversing the orientation of rotation, reverses the sign of this pressure. This response to the rotation is not particularly intuitive.

Materials with significant odd viscosity, be they solids or liquids, can support non-dissipating chiral viscosity waves with a quadratic dispersion. The reflection from a boundary of these waves obeys a different rule than the reflection of acoustic waves. In addition, such materials can function as circular polarizers for ordinary acoustic waves. At the same time, viscosity waves can be quite elusive. In particular we shall show that there are no viscosity waves in isotropic and incompressible media. In particular, the Hall system, being isotropic and incompressible, does not support viscosity waves.

A dimensionless number that gives a measure of the importance of viscosity relative to elasticity is

$$\varepsilon = \frac{\nu\omega}{c_s^2} \tag{3}$$

where ν is the kinematic viscosity, ω the frequency, and c_s the velocity of sound. Odd viscosity is always unimportant relative to elasticity at low frequencies. For the dissipating viscosity, and also for the odd viscosity in the Hall effect, the kinematic viscosity is of order 1 in cgs units. Then, since c_s is typically of order 10^5 in cgs, ε is of order unity for ω of order GHz.

As we shall see, the generalized Navier–Stokes equation that allows for odd viscosity preserves the basic properties of fluid dynamics of the ordinary Navier–Stokes equation: Kelvin theorem and Bernoulli law generalize to non zero odd viscosity.

2. ODD VISCOSITY

Consider a hypothetical, homogeneous and ideal Newtonian fluid. The stress due to viscosity is:

$$\sigma^{v}_{ij} = \eta_{ijkl} \dot{u}_{kl} \tag{4}$$

where $u_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the strain, $\dot{u}_{jk}$ is the strain rate, η a (constant) viscosity tensor and σ the stress. By general principles,[1] the viscosity tensor, η_{ijkl}, is symmetric under $i \leftrightarrow j$, and $k \leftrightarrow l$. One can always write $\eta = \eta^S + \eta^A$ where η^A is anti symmetric under $\{ij\} \leftrightarrow \{kl\}$ and η^S is symmetric under $\{ij\} \leftrightarrow \{kl\}$. By Onsager relation[2] the anti-symmetric part is odd under time reversal and the symmetric part is even, e.g., with B an external magnetic field;

$$\eta^S(B) = \eta^S(-B), \qquad \eta^A(B) = -\eta^A(-B) \tag{5}$$

So, for $\eta^A \neq 0$ time reversal must be broken. One can ask if odd viscosity is not just a way of mascarading, say, an external magnetic field. The answer to this is no. This can be seen by a counting argument. A (constant) magnetic field has three components while the space of anti-symmetric tensors η^A is 15 dimensional in three dimensions.

2.1. Viscosity in Two Dimensions

In two dimensions there is a natural basis for representing real 4th rank tensor which are symmetric in pairs:[2]

$$\eta = \sum \eta_{ab} \sigma^a \otimes \sigma^b, \qquad a, b \in \{0, 1, 3\} \tag{6}$$

Here σ^a are the Pauli matrices with σ^0 the unit matrix. σ^2, which is conventionally imaginary, is not used. In components:

$$\eta_{ijkl} = \sum \eta_{ab} \sigma^a_{ij} \sigma^b_{kl} \qquad i, j \in \{0, 1, 3\} \tag{7}$$

[2] This basis was suggested by the referee of this paper.

The space of such tensors is nine dimensional, and its splits to a six dimensional even part and three dimensional odd part. The odd part takes the form

$$\eta^A = \sum_{a \neq b} \eta^A_{ab}(\sigma^a \otimes \sigma^b - \sigma^a \otimes \sigma^b) \qquad i, j \in \{0, 1, 3\} \tag{8}$$

To see what happens in isotropic media, recall that $i\sigma^2$ is the generator of rotations in two dimension, and it anticommutes with σ^a for $a = 1, 3$ and commutes for $a = 0$. Hence $\sigma^2 \otimes \sigma^2$ commutes with $\sigma^a \otimes \sigma^b$ if $a, b = 1, 3$ and $a = b = 0$. The isotropic symmetric part is two dimensional and is characterized by two viscosity coefficients:

$$\eta^S = \eta^s(\sigma^1 \otimes \sigma^3 + \sigma^3 \otimes \sigma^1) + \zeta \sigma^0 \otimes \sigma^0 \tag{9}$$

The odd isotropic part is one dimensional and is of the form

$$\eta^A = \eta^a(\sigma^1 \otimes \sigma^3 - \sigma^3 \otimes \sigma^1) \tag{10}$$

The non-zero components are determined by

$$\eta^A_{1211} = -\eta^A_{1222} = \eta^a \tag{11}$$

It is sufficient that a two dimensional medium is invariant under rotation by $\pi/4$ for η to have this form.

As a consequence of this, isotropic two dimensional media are characterized, in general, by *three* coefficients of viscosity: two for the even part and the third for the odd viscosity η^a. This is in contrast with a claim made in ref. 3 (p. 45–46), and in ref. 6, Section 15, that isotropy alone implies that the viscosity tensor has two coefficients of viscosity.

The stress σ^v associated with the viscosity of an isotropic medium, is

$$\begin{aligned} \sigma^v_{ij} &= \eta^s(\dot{u}_{ij} + \dot{u}_{ji}) + \zeta\, \delta_{ij}\dot{u}_{kk} \\ &\quad - 2\eta^a(\delta_{i1}\,\delta_{j1} - \delta_{i2}\,\delta_{j2})\,\dot{u}_{12} + \eta^a(\delta_{i1}\,\delta_{j2} + \delta_{i2}\,\delta_{j1})(\dot{u}_{11} - \dot{u}_{22}) \end{aligned} \tag{12}$$

In an incompressible fluid $\dot{u}_{kk} = 0$ and the stress becomes independent of ζ, the second viscosity coefficient. Because of this ζ plays no role in the Navier–Stokes equation for incompressible fluids. Incompressible and isotropic fluids in two dimensions with broken time reversal are characterized by two viscosity coefficients one for the odd part, η^a, and one for the even part η^s.

2.2. Viscosity in Three Dimensions

In three dimensions isotropy implies that $\eta^A = 0$. Three dimensional isotropic media are characterized by only *two* coefficients of viscosity, and both of these are associated with the even part of the viscosity tensor. Isotropic and incompresible media are characterized by a single dissipative viscosity coefficient.

The vanishing of the odd viscosity for isotropic tensors can be seen by the following elementary argument, which I owe to L. Sadun. Vectors in three dimensions are associated with the **1** representation of the rotation group. 2-tensors are identified with

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{0}$$

representation of the rotation group. The vector representation **1** on the right hand side, is associated with pseudo vectors and so with the antisymmetric 2-tensors. The $\mathbf{2} \oplus \mathbf{0}$ is the six dimensional representation associated with the symmetric 2-tensors. This shows that there is one isotropic 2-tensor in three dimensions, namely the identity. Continuing in this vein, tensors t_{ijkl} with the symmetry $i \leftrightarrow j$, and $k \leftrightarrow l$ are identified with

$$(\mathbf{2} \oplus \mathbf{0})^2 = \mathbf{4} \oplus \mathbf{3} \oplus 3 \cdot \mathbf{2} \oplus \mathbf{1} \oplus 2 \cdot \mathbf{0} \tag{13}$$

This shows that the space of isotropic 4-tensors with the symmetry $i \leftrightarrow j$, and $k \leftrightarrow l$ is 2-dimensional. The isotropic 4-tensors t^S make a two dimensional family, given by

$$t^S_{ijkl} = t_1 \, \delta_{ij} \, \delta_{kl} + t_2(\delta_{ik} \, \delta_{jl} + \delta_{il} \, \delta_{jk}) \tag{14}$$

this leaves nothing for the isotropic antisymmetric 4-tensor and so $\eta^A = 0$.

The number of independent components of the odd and even part of the viscosity tensor is given in Table I.

Table I

	Even		Odd	
Dimension	general	isotropic	general	isotropic
2	9	2	3	1
3	21	2	15	0

3. VISCOSITY WAVES

Consider a hypothetical ideal three dimensional medium where the stress tensor is dominated by the odd viscosity. For small oscillations the equation of motion is [1]:

$$\rho \ddot{u}_i = \partial_j \sigma^v_{ij} \tag{15}$$

where ρ is the (mass) density, u_j the (cartesian) jth component of the displacement (assumed small), and σ^v, the stress tensor, depends linearly on the strain rate. Linear elasticity is mathematically trivial in the sense that solving the PDE reduces to a problem in linear algebra. In a homogeneous systems ρ is a constant so one can, without loss, set $\rho = 1$ (by absorbing ρ in σ.) For a medium with odd viscosity Newton equation gives a linear PDE which is second order in space and time:

$$\ddot{u}_i = \eta^A_{ijkl} \dot{u}_{l,\,jk} \tag{16}$$

This equation is *not* just an alternative way of describing the Lorentz force. The corresponding Newton–Lorentz equation:

$$\ddot{\mathbf{u}} + \mathbf{B} \wedge \dot{\mathbf{u}} = 0 \tag{17}$$

is not a PDE.

Newton equation for viscosity waves admits an integral and by choosing the constant of integration appropriately, reads:

$$\dot{u}_i = \eta^A_{ijkl} u_{l,\,jk} \tag{18}$$

This equation is first order in time and second order in space. Because η^A is antisymmetric it has the same character as Schrödinger equation.[3] But it is a classical equation in the sense that the wave function is directly observable.

Consider plane waves solutions to this equation with wave vector $\mathbf{k}$. Let $\mathbf{\Omega}(\mathbf{k})$ be the (pseudo) vector with components that are quadratic forms in the coordinates k_j:

$$\Omega_i(\mathbf{k}) = \varepsilon_{ijk} \eta^A_{j\alpha\beta k} k_\alpha k_\beta \tag{19}$$

Newton law for a plane wave propagating in the $\mathbf{k}$ direction can be written as:

$$\dot{\mathbf{u}} = \mathbf{\Omega}(\mathbf{k}) \wedge \mathbf{u} \tag{20}$$

[3] In real coordinates, with $\psi_1 = \mathrm{Re}\,\psi$ and $\psi_2 = \mathrm{Im}\,\psi$, Schrödinger equation for a free particle takes the form $\dot{\psi}_i = \varepsilon_{ij} \psi_{j,\,kk}$.

From this equation it is clear that the viscosity wave in three dimensions is chiral and circularly polarized in the plane perpendicular to the vector $\boldsymbol{\Omega}(\mathbf{k})$ and has quadratic dispersion. Recall that ordinary sounds waves have linear dispersion, but, rods and plates also allow for elastic modes with quadratic dispersion.[(1)] The equation for viscosity waves, Eq. (20), is formally identical to the Landau Lifshitz equation for magnons.[4]

3.1. Incompressible Media

Consider an incompressible medium where $\nabla \cdot \mathbf{u} = 0$. For a plane wave this means that $\boldsymbol{\Omega}$ is parallel to $\mathbf{k}$, that is:

$$\boldsymbol{\Omega} \wedge \mathbf{k} = 0, \Leftrightarrow 0 = \eta^A_{ijkl} k_j k_k k_l \tag{21}$$

For a viscosity wave propagating in the 1 direction

$$\boldsymbol{\Omega}(\mathbf{k}) = k^2 \begin{pmatrix} \eta^A_{2113} \\ \eta^A_{3111} \\ \eta^A_{1112} \end{pmatrix} \tag{22}$$

From this it follows that in an incompressible medium either $\eta^A_{i111} = 0$ for all i or $\mathbf{u} = 0$ identically. This leads to:

Proposition 3.1. There are no odd viscosity waves in isotropic and incompressible fluids. In particular, there are no viscosity waves in a two dimensional quantum Hall fluid.

This is so because in two dimensions isotropic fluids the non zero component of the isotropic odd viscosity are $\eta^a = -\eta^A_{1222} = \eta^A_{2111}$. For an isotropic fluid we can choose the 1 axis to coincide with the wave vector, and then incompressibility says that either $\eta^a = \eta^A_{2111} = 0$ or the wave has zero amplitude.[5] In three dimensions, there are no viscosity waves because the odd viscosity vanishes.

3.2. The Energy Flux of Viscosity Waves

In the case $\eta^S = 0$ one can define a conserved current associated with conservation of energy. The kinetic energy density:

$$\mathscr{E} = \tfrac{1}{2}(\mathrm{Re}\, \dot{\mathbf{u}})^2 \tag{23}$$

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[4] I thank Dr. M. Milovanovic for reminding me of this fact.

[5] The result, and proof, works also for a two dimensional planar medium embedded in three dimensions.

satisfies a conservation law:

$$\partial_t \mathscr{E} = \partial_i J_i \tag{24}$$

where the energy flux J is

$$J_i = \eta^A_{ijkl} \operatorname{Re}(\dot u_j) \operatorname{Re}(\dot u)^{kl} = \sigma^v_{ij} \operatorname{Re}(\dot u_j) \tag{25}$$

It follows that odd viscosity carries energy flux. There is a generalization of this energy flux to media where the stress also has elastic part.

3.3. Dissipative Shear Waves

It is instructive to contrast viscosity waves with the strongly dissipative shear waves.[6] For shear waves $\eta^S \neq 0$ and $\eta^A = 0$, and the dispersion curves are *pure imaginary*

$$i\omega = -\eta^S k^2, \qquad \eta^S, \omega > 0 \qquad \operatorname{Im} k > 0 \tag{26}$$

Shear waves decay rapidly, by a factor of about $\exp(2\pi) \approx 540$ in one period.[1] If one now consider a mixed situation where η^S and η^A are of the same magnitude then the attenuation is much weaker—of order $\exp(2\pi \tan(\pi/8)) \approx 13$.

4. SCATTERING OF VISCOSITY WAVES

Using the rules of geometric acoustics it is straightforward to find the laws of reflection and refraction of an acoustic wave (in a medium where $\eta = 0$) from an interface with an odd viscous medium (where the stress is dominated by η^A). If we let y be the axis separating the two media, then k_2 is conserved and so is the frequency ω. Let θ be the angle of incidence, $\cos(\theta) = \hat{x} \cdot \hat{k}_i$, of an acoustic wave with sound velocity c_s and ϕ the angle of the transmitted viscosity wave, $\cos(\phi) = \hat{x} \cdot \hat{k}_t$. Then a little geometry shows that

$$\sin^2 \phi = \varepsilon \sin^2 \theta, \qquad \varepsilon = \frac{\eta\omega}{\rho c_s^2} \tag{27}$$

We see that $\varepsilon^{-1/2}$ plays the role of the index of reflection. Since ε is small at low frequencies, the index of reflection is large, and the transmitted beam is essentially normal to the interface. Since the index depends on ω the reflection is dispersive.

4.1. Reflection from Empty Space

Consider the reflection of ideal viscosity waves in *two dimensional isotropic and compressible* medium from an interface with a vacuum which we take to be the line $x = 0$. A viscosity waves is

$$\mathbf{u}(z) = \begin{pmatrix} 1 \\ i \end{pmatrix} z, \qquad z = \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t) \tag{28}$$

where $\omega = \eta^a k^2$. The waves is circularly polarized in the plane and changes polarization from longitudinal to transversal. The boundary conditions at $x = 0$ are

$$\sigma_{11} = \sigma_{12} = 0 \tag{29}$$

For this wave:

$$\sigma_{11}(\mathbf{u}) = -\eta_{1112}\bar{k}, \qquad \sigma_{12}(\mathbf{u}) = -i\eta_{1112}\bar{k}; \qquad k = k_1 + ik_2, \quad \bar{k} = k_1 - ik_2 \tag{30}$$

Let $k = (k_1, k_2)$ be the incident wave, $\tilde{k} = (-k_1, k_2)$ be the reflected wave; and $\tilde{z} = \exp i(\tilde{\mathbf{k}} \cdot \mathbf{x} - \omega t)$. $\mathbf{u}(z)$ is the incident wave (from the left) and $\mathbf{u}(\tilde{z})$ the reflected wave and $R(k)$ the reflection amplitude. The wave on the left is:

$$\mathbf{u}(z) + R(k)\,\mathbf{u}(\tilde{z}), \qquad x < 0 \tag{31}$$

Substituting in the boundary conditions, using Eq. (30) and linearity, one finds

$$R(k) = \frac{k}{\bar{k}} \tag{32}$$

The phase of the reflected wave has an interesting dependence on the direction of incidence: It goes from 1 for normal reflection to -1 for grazing reflection. This is quite unlike the reflection of say longitudinal sound waves in an incompressible liquid where the reflection amplitude is independent of the direction of incidence.

4.2. Odd Viscosity as Polarizer

Consider a wave propagating in the x direction, of a three dimensional incompressible medium, so that to the left there is an isotropic elastic medium with velocities of sound c_t and c_l. To the right is a viscous (possibly incompressible) medium. The $y-z$ plane separates the two media. Incompressibility says that $\mathbf{\Omega}(\mathbf{k})$ points in the x direction. Let

$$\mathbf{e}_+ = (0, 1, i), \qquad \mathbf{e}_- = (0, 1, -i) \tag{33}$$

denote the two basic vectors of circular polarization. Consider scattering of a transverse wave with positive chirality from the $y-z$ interface. The incoming and reflected waves in the elastic medium are

$$\mathbf{e}_+(\exp ik(x - c_t t) + r \exp ik(-x - c_t t)), \qquad x < 0 \tag{34}$$

The transmitted wave (necessarily with positive chirality) is:

$$t\mathbf{e}_+ \exp i(\tilde{k}x - \omega t), \qquad x > 0 \tag{35}$$

Where $\omega = kc_t = |\mathbf{\Omega}(\tilde{\mathbf{k}})|$. The basic boundary conditions matches σ^L_{1j} on the left with σ^R_{1j} on the right. One checks that $\sigma^L_{11} = \sigma^R_{11} = 0$ and that $i\sigma^R_{12} = \sigma^R_{13}$. $i\sigma^L_{12} = \sigma^L_{13}$ is automatically satisfied (confirming the anzatz that there is no reflected wave with flipped chirality). The remaining equation, $\sigma^L_{12} = \sigma^R_{12}$, gives

$$1 - r = \varepsilon t, \qquad \varepsilon = \frac{\eta_{2113}\omega}{\rho c_t^2} > 0 \tag{36}$$

Continuity of the wave u on the boundary gives two equations but only one new: $1 + r = t$. From these I finally get:

$$t = \frac{2}{1+\varepsilon}, \qquad r = \frac{1-\varepsilon}{1+\varepsilon}, \qquad \varepsilon = \frac{\eta_{2113}\omega}{\rho c_t^2} \tag{37}$$

Note that $|r| \leqslant 1$ as it should: The reflected wave always has smaller amplitude than the incident wave. However, for t one gets an unusual behavior: $0 \leqslant t \leqslant 2$ and the amplitude t is *maximal* for $\varepsilon = 0$ when the *amplitude* of the transmitted wave is twice as large as that of the incident wave! This surprise is mitigated when one notices that for the energy flux in the x direction, J_1, the dependence on ε is $\varepsilon/(1+\varepsilon)^2$, which gives maximal flux at $\varepsilon = 1$ as one expects, since in this case $r = 0$. The wave is perfectly transmitted.

Consider now the scattering of a viscosity wave so that the incident wave has *negative* chirality. The most general transverse wave in the elastic medium with the given incident wave is:

$$r\mathbf{e}_+ \exp ik(-x - c_t t) + \mathbf{e}_-(\exp ik(x - c_t t) + \tilde{r} \exp ik(-x - c_t t)) \tag{38}$$

while the wave in the viscous medium is as before. Matching σ^L_{1j} with σ^R_{1j} one finds that as before $\sigma^L_{11}=\sigma^R_{11}=0$ and $i\sigma^R_{12}=\sigma^R_{13}$. Writing out $i\sigma^L_{12}=\sigma^L_{13}$ gives: $\tilde{r}=1$. The remaining equation, $\sigma^L_{12}=\sigma^R_{12}$, gives

$$r=\varepsilon t \tag{39}$$

This is all that follows from the continuity of the three stress components. Imposing continuity of the wave on the boundary gives *two* new equations:

$$r=t, \qquad 1+\tilde{r}=0 \tag{40}$$

The second equation is in conflict with the equation for the continuity of the stress.[6] The way out is *not* to require continuity of the wave, but instead continuity of the energy flux. This holds if the flux vanishes at the surface and sets $r=t=0$. The incident wave with the wrong polarization is totally reflected. We see that *anisotropic incompressible media with odd viscosity act as circular polarizers.*

5. NAVIER–STOKES EQUATION

Consider the Navier–Stokes equation, for homogeneous ($\rho=1$), isotropic and incompressible fluid in two dimensions—the one dimension where odd viscosity is compatible with isotropy.

With broken time reversal, the general Navier–Stokes equation for the pressure p and the velocity field $\mathbf{v}$, is the obvious generalization of the standard Navier–Stokes equation:[(3)]

$$\partial_t \mathbf{v}+(\mathbf{v}\cdot\nabla)\,\mathbf{v}=-\nabla p+\eta^s\,\Delta\mathbf{v}-\eta^a\,\Delta\mathbf{v}^*; \qquad \nabla\cdot\mathbf{v}=0 \tag{41}$$

The dual is defined, as usual, by

$$v_i^*=\varepsilon_{ij}v_j \tag{42}$$

Incompressibility is implemented by introducing a stream function and the equations can be alternatively written as four equations for the four fields $(p, \psi, \mathbf{v})$:

$$\partial_t \mathbf{v}+(\mathbf{v}\cdot\nabla)\,\mathbf{v}=-\nabla(p-\eta^a\xi)+\eta^s\,\Delta\mathbf{v}; \qquad \mathbf{v}=(\nabla\psi)^*, \qquad \xi=-\Delta\psi \tag{43}$$

[6] This is the generic situation in scattering of acoustic and viscosity waves. That is, in general imposing continuity of the wave across the interface gives an overdetermined system. In this sense, the previous example is exceptional in that both the stress and the wave turned out to be continuous.

Taking curl the equation for the vorticity is

$$\partial_t \xi + (\mathbf{v} \cdot \nabla)\, \xi = \eta^s\, \Delta \xi \tag{44}$$

Vorticity is be generated only by the symmetric (dissipative) part of the viscosity tensor. The odd viscosity does not generate vorticity.

5.1. Bernoulli Law

Consider incompressible fluid in two dimensions, with $\eta^s = 0$, in steady state. The equation of motion is:

$$(\mathbf{v} \cdot \nabla)\, \mathbf{v} = -\nabla(p - \eta^a \xi) \tag{45}$$

Since

$$\tfrac{1}{2} \nabla(\mathbf{v}^2) = (\mathbf{v} \cdot \nabla)\, \mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla(p - \eta^a \xi) + \mathbf{v} \times (\nabla \times \mathbf{v}) \tag{46}$$

one gets, by integrating along a streamline, that

$$\tfrac{1}{2} \mathbf{v}^2 + p - \eta^a \xi = \text{const.} \tag{47}$$

This looks at first like an interesting generalization of Bernoulli's law to odd viscosity with the pleasant feature that the vorticity (weighted by the odd viscosity) plays a role of (signed) kinetic energy. However, as vorticity is conserved along a streamline, it is actually precisely equivalent to Bernoulli's law.

5.2. Stokes Equation

The limit of small Reynolds number is described by Stokes equation. This is the case where normally viscosity dominates. Stokes equation is linear, so this is also the easy limit. In the present context, with broken time reversal, and for two dimensional isotropic and incompressible fluid:

$$\partial_t \mathbf{v} = -\nabla(p - \eta^a \xi) + \eta^s\, \Delta \mathbf{v}; \qquad \mathbf{v} = (\nabla \psi)^* \tag{48}$$

Taking curl and div we get:

$$\partial_t \xi = \eta^s\, \Delta \xi, \qquad 0 = -\Delta(p - \eta^a \xi) \tag{49}$$

In a steady state, if $\eta^S \neq 0$, both the pressure and the vorticity must be harmonic functions and ψ bi-harmonic.

5.3. A Rotating Disc

Consider a rigid disc which is rotating with constant (and small) angular velocity Ω in a fluid subject to no-slip boundary conditions. Stokes equation can be solved explicitly in this case[7, 8]. I will show that while the ordinary viscosity applies a torque that resists the rotation of the disc, the odd viscosity leads to a pressure on the disc which is proportional to the angular velocity. The pressure can be either positive or negative, depending on the sense of rotation.

Written in complex notation Stokes equation read:

$$\bar{\partial}(p + i\eta\xi) = 0, \qquad \eta = \eta^s + i\eta^a \tag{50}$$

Since ψ is bi-harmonic and real

$$-2i\psi = z\,\overline{a(z)} - \bar{z}a(z) + b(z) - \overline{b(z)} \tag{51}$$

Let $v = v_1 + iv_2$. Using

$$v = -2i\,\bar{\partial}\psi, \qquad \xi = -2i\,\partial v \tag{52}$$

one finds

$$v = -a(z) + z\,\overline{a'(z)} - \overline{b'(z)}, \qquad \xi = 2i(\overline{a'(z)} - a'(z)) \tag{53}$$

Since the velocities at infinity are finite, one must haves[7]

$$a(z) = \sum_{j=0}^{\infty} \frac{a_j}{z^j}, \qquad b(z) = b\log z + \sum_{j=-1}^{\infty} \frac{b_j}{z^j} \tag{54}$$

On the surface of the circle $\bar{z}z = 1$ and one can trade $\bar{z}$ for an inverse power of z, so:

$$i\Omega z = -\sum_{j=0}^{\infty} \frac{(j-1)\,\bar{b}_{j-1} + a_j}{z^j} - \sum_{j=3}^{\infty} (j-2)\,\bar{a}_{j-2} z^j - \bar{b}z, \qquad |z| = 1 \tag{55}$$

Equating coefficients we get

$$i\Omega = -\bar{b} \tag{56}$$

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[7] One can actually allow a_{-1} real. One can not allow a log term in $a(z)$ if one is interested in continuous solutions for the velocity field.

All other coefficients are zero. That is

$$v = -i\,\frac{\Omega}{\bar{z}}, \qquad \xi = 0 \tag{57}$$

It then also follows that p is a constant. Using the velocity field one finds the strain on the surface of the circle:

$$\dot{u}_{11} = -\Omega\sin(2\theta), \qquad \dot{u}_{12} = \Omega\cos(2\theta), \qquad \dot{u}_{22} = \Omega\sin(2\theta) \tag{58}$$

Here $x = |z| \cos\theta$. Let us write the stress as $\sigma = \sigma^s + \sigma^a$ where the first piece is due to the dissipative viscosity and the second is due to the odd viscosity. Then

$$\sigma^s_{ij} = 2\eta^s \dot{u}_{ij} \tag{59}$$

And

$$\sigma^a_{11} = 2\eta^a\Omega\cos(2\theta), \qquad \sigma^a_{12} = 2\eta^a\Omega\sin(2\theta), \qquad \sigma^a_{11} = -2\eta^a\Omega\cos(2\theta) \tag{60}$$

Let n and t denote the unit vectors normal and tangent to the disc. A calculation gives

$$\sigma^s_{nn} = 2\eta^a\Omega, \qquad \sigma^s_{tn} = 2\eta^s\Omega \tag{61}$$

This says that the symmetric viscosity resists the rotation by applying a torque on the rotating circle. The odd viscosity applies no torque, but instead, a normal pressure on the circle, which is proportional to the rate of rotation.

Remark. One could have also asked what is the effect of odd viscosity on the drag and lift (the Magnus force). Unfortunately, for an incompressible, viscous fluid in two dimensions the question is moot: Stokes equation in two dimension does not admit steady state solutions that describe a moving disc with no slip boundary conditions in a fluid that is at rest at infinity. Two dimensions behave like one dimension and unlike three dimensions. This feature of the Stokes equation is known in classical fluid mechanics as the Whitehead paradox.

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REFERENCES

1. L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, Third Edition, by E. M. Lifshitz, A. M. Kosevitch, and L. P. Pitaevskii (Pergamon, Oxford, 1986).
2. L. Onsager, Reciprocal relations in Irreversible Processes, *Phys. Rev. B* **37**:405, **38**:2265 (1931).
3. A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics* (Springer, 1979).
4. J. E. Avron, R. Seiler, and P. Zograf, Viscosity of quantum hall fluids, *Phys. Rev. Lett.* **75**:697–700 (1995).
5. P. Levay, Berry phases for Landaus Hamiltonians on deformed tori, *J. Math. Phys.* **36**:2792 (1995).
6. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1959).
7. N. I. Muskhelishvili, *Some Basic Problems in the Mathematical Theory of Elasticity* (Noordhof, 1953).
8. A. Shapere and F. Wilczek, *Geometric Phases in Physics* (World Scientific, 1989).
9. D. Vollhardt and P. Wölfle, *The Superfluid Phases of* He^3 (Taylor and Francis, 1990).